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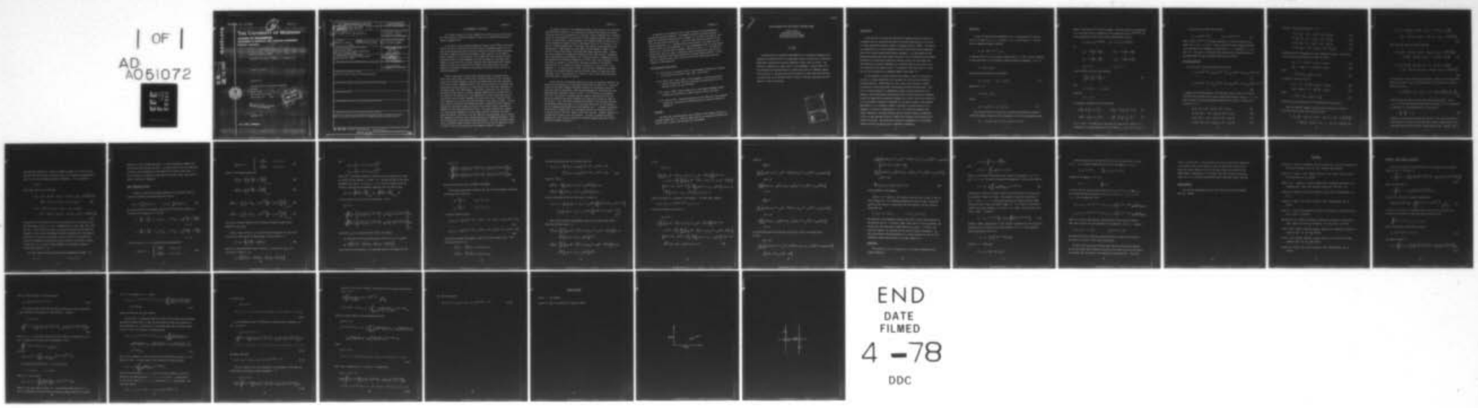

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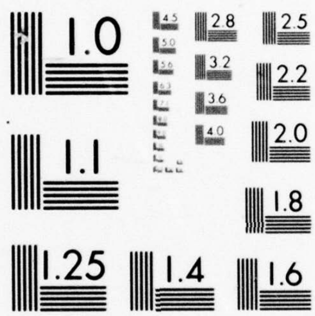
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COLLEGE OF ENGINEERING

DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

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Thomas B.A. Senior

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This interim scientific report summarizes the work carried out under the Air Force Office of Scientific Research Grant No 77-3188 during the year ended 31 Dec 77.

ELECTROMAGNETIC SCATTERING

This interim scientific report summarizes the work carried out under the Air Force Office of Scientific Research Grant No. 77-3188(A) during the year ended 31 December 1977.

It has been a year of steady progress on several fronts. One of the major efforts has been in connection with the design of broadband absorbers to reduce the backscattering cross section of wing-like structures when illuminated at close to edge-on incidence with magnetic vector parallel to the edge. Under a previous Air Force Contract it was shown that thin coatings could be simulated using a surface impedance, and that the materials available could produce a significant cross section reduction over broad aspect and frequency ranges, but the time available did not permit the detailed analytical and numerical investigation necessary to determine the maximum cross section reduction attainable.

Such an investigation has now been performed under the present Grant. Using a 30° angle ogival cylinder with a maximum allowed value for the magnitude of the surface impedance, analyses and computations have been performed to find the surface impedance profiles that are most effective in reducing the backscattering cross section for H polarization, with particular reference to an aspect range ± 30° about edge-on. Analytical expressions for the fields diffracted by the edge of a uniform impedance wedge and by a discontinuity in the value or first derivative of an impedance on a plane surface were used to specify the (local) surface impedance to minimize the direct scattering from any singularity, as at the front and rear edges of the ogival cylinder. Various trial profiles were then constructed and used in conjunction with a computer program for the direct digital solution of the surface integral equations to compute the resulting backscattered fields. From data obtained at a sequence of closely spaced frequencies, it was found possible to separate the front edge contribution from that of the rear edge and traveling waves, and so choose the impedance to minimize each, thereby assuring a broadband performance. Cross section reductions of more than 20 dB were achieved. The work has been written up as a technical report (Senior, 1977) which will be sent for approval of publication as soon as the drafting of the numerous figures is complete.

Our general studies of the impedance boundary condition have also continued with particular emphasis on its application to bodies having edge and corners. With some bodies the introduction of a non-zero surface impedance has a profound effect on the character of the solution of the scattering problem, and a case in point is a plane wave at skew incidence on an impedance wedge of open angle $\pi/2$. If the impedance of either surface of the wedge is zero or if the wave is incident in a plane perpendicular to the edge, the solution can be obtained using Maliuzhinets' (1959) method or, more simply, by the method of images, and consists of four plane waves: the primary and three reflected waves. However, for skew incidence on a wedge both surfaces of which have non-zero impedances, these same methods are applicable only if the (anisotropic) impedances satisfy the compatibility relation of Dybdal et al (1971), and if this condition is not satisfied, no method is available for the exact solution of the problem. Nevertheless, it is possible to obtain a solution as a Taylor series in the impedance η. As shown by Senior (1978), for arbitrary impedances the angular spectrum is no longer discrete, and the solution contains an edge wave whose amplitude is $O(\eta^2)$ and whose form is similar to that of the Kottler edge wave associated with a 'black' half plane.

In the area of low frequency scattering we have continued our numerical and analytical investigations of the general polarizability tensor $\bar{\bar{X}}(\tau)$. Numerical values of the tensor elements have been obtained for rectangular dielectric particles (Herrick and Senior, 1977) and the mathematical formulation has now been applied to hexagonal crystals as well. As in the case of rectangular particles, the tensor is diagonal and the integral equations which must be solved to determine the tensor elements are singular only along the edges of the crystal. This simplifies the numerical solution of the integral equations by the moment method and again it is found that the kernels of the equations can be integrated analytically over each sampling cell. A computer program has been written to compute $\bar{\bar{X}}(\tau)$ for a hexagonal crystal for arbitrary values of τ and the ratio of the axial length to the cross section. However, analysis of the numerical results is not yet complete. In order to compute the tensor elements for lossy dielectric materials, modified programs have been written which allow complex values of τ.

In addition to the numerical work, several aspects of the mathematical formulation have been examined. The symmetry relations which were assumed among the potential functions (see for example (11) of Herrick and Senior, 1977) have now been proved analytically. It can be shown that the solutions of the integral equations, i.e., the potentials on the surface of the body, are odd (even) functions of the coordinate x_i if the inhomogeneous term is an odd (even) function of x_i and the body is symmetric about the plane $x_i = 0$. We have also proved that, even for a body of arbitrary shape, the potentials given by the integral equations satisfy the required zero induced charge condition. In the case of a symmetric body, such a proof can easily be constructed using the symmetry of the potentials.

Grant-Supported Publications

D.F. Herrick and T.B.A.Senior (1977), "Low frequency scattering by rectangular dielectric particles", Appl. Phys. 13, 175-183.

T.B.A. Senior and H. Weil (1977), "Electromagnetic scattering and absorption by thin-walled dielectric cylinders with application to ice crystals", Appl. Opt. 16, 2979-2985.

T.B.A. Senior (1978), "Skew incidence on a right-angled impedance wedge", accepted for publication in Radio Science; preprint enclosed.

T.B.A. Senior (1977), "Analyses pertaining to the reduction of non-specular scattering", University of Michigan Radiation Laboratory Report No. 015224-1-T.

Personnel

The Grant has provided partial salary support of the project director, a graduate student (Mr. D.F. Herrick, whose work on low frequency scattering will form his Ph.D. thesis), and two part-time programmers.

SKEW INCIDENCE ON A RIGHT-ANGLED IMPEDANCE WEDGE

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ABSTRACT

A problem which is relevant to propagation in a rectangular waveguide with imperfectly conducting walls is a plane wave at skew incidence on a right-angled wedge whose surfaces have non-zero impedances either scalar or tensor. The solution is obtained as a power series in the impedance η through terms $O(\eta^2)$. Its properties are discussed and it is shown that unless an impedance compatibility relation is satisfied the solution contains an edge wave whose amplitude is proportional to η^2 and whose form is similar to the wave emanating from the edge of a 'black' half plane.

INTRODUCTION

In the analysis of scattering problems the impedance boundary condition is a valuable tool for simulating the effects of surface imperfections in the form of finite conductivity (Senior, 1959a) or roughness (Senior, 1959b). The effects are represented via a surface impedance, and the results have proved meaningful even in the case of bodies having corners and edges. For some geometries the scattering problem can now be solved using only a trivial extension of the method applicable to the perfectly conducting (smooth) surface, but with others the introduction of a non-zero impedance has a profound effect on the analysis and the character of the solution itself. An example of the latter type is a plane wave at skew incidence on an impedance wedge of open angle $\pi/2$.

If the impedance of either surface of the wedge is zero or if the wave is incident in a plane perpendicular to the edge, the solution can be obtained using Maliuzhinets' (1959) method or, more simply, by the method of images, and consists of four plane waves: the primary and three reflected waves. However, for skew incidence on a wedge both surfaces of which have non-zero impedances, these same methods are applicable only if the (anisotropic) impedances satisfy the compatibility relation of Dybdal et al (1971), and if this condition is not satisfied, no method is available for the exact solution of the problem. Nevertheless, it is possible to obtain a solution as a Taylor series in the impedance η and this is determined to $O(\eta^2)$. For arbitrary impedances the angular spectrum is no longer discrete and the solution contains an edge wave similar to that obtained by Kottler (1923) in his analysis of diffraction by a 'black' half plane. The results are relevant to propagation in a rectangular waveguide having two adjacent walls imperfectly conducting.

FORMULATION

In terms of the Cartesian coordinates (x, y, z) the equations of the two faces of the wedge are $y = 0$, $x \geq 0$ and $x = 0$, $y \geq 0$ (see Figure 1). At each face the impedance boundary condition

$$\underline{E} - (\hat{n} \cdot \underline{E})\hat{n} = Z \, \bar{\bar{\eta}} \cdot \hat{n} \times \underline{H}$$

is imposed, where $\hat{n}$ is a unit vector outward normal, Z is the intrinsic impedance of free space and $\bar{\bar{\eta}}$ is an anisotropic (tensor) normalized impedance. On $y = 0$

$$\bar{\bar{\eta}} = \eta_1 \, \hat{x}\hat{x} + \eta_2 \, \hat{z}\hat{z}$$

and the boundary conditions are therefore

$$E_x = \eta_1 \, Z \, H_z \, , \qquad E_z = - \eta_2 \, Z \, H_x \, , \tag{1}$$

whereas on $x = 0$

$$\bar{\bar{\eta}} = \eta_3 \, \hat{y}\hat{y} + \eta_4 \, \hat{z}\hat{z}$$

giving

$$E_y = - \eta_3 \, Z \, H_z \, , \quad E_z = \eta_4 \, Z \, H_y \, . \tag{2}$$

The incident field is a plane wave of arbitrary polarization incident in a direction making an angle β with the negative z axis, with propagation vector

$$\underline{k} = - k \, (\hat{x} \cos \alpha \cos \beta + \hat{y} \sin \alpha \cos \beta + \hat{z} \sin \beta)$$

where a time factor $e^{-i\omega t}$ has been assumed. Since the surfaces are independent of z, the total field will depend on z only through the factor $e^{-ikz\sin\beta}$, and can be expressed in terms of the single component Hertz vectors

$$\underline{\Pi} = \hat{z}\, U\,(x, y)\, e^{-ikz\sin\beta}, \qquad \underline{\Pi}^* = \hat{z}\, V\,(x, y)\, e^{-ikz\sin\beta}$$

as

$$E_x = -\, ih \frac{\partial U}{\partial x} + ik \frac{\partial V}{\partial y}\ , \qquad Z\, H_x = -\, ih \frac{\partial V}{\partial x} - ik \frac{\partial U}{\partial y}\ ,$$

$$E_y = -\, ih \frac{\partial U}{\partial y} - ik \frac{\partial V}{\partial x}\ , \qquad Z\, H_y = -\, ih \frac{\partial V}{\partial y} + ik \frac{\partial U}{\partial x}\ ,$$

$$E_z = \lambda^2\, U\ , \qquad Z\, H_z = \lambda^2\, U\ .$$

U and V satisfy the scalar wave equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \lambda^2\right)\begin{matrix}U\\V\end{matrix} = 0 \tag{3}$$

and

$$h = k\,\sin\,\beta, \qquad \lambda = k\,\cos\,\beta$$

implying

$$h^2 + \lambda^2 = k^2\ .$$

The boundary conditions on U and V are then

$$h\,\frac{\partial U}{\partial x} = \left(k\,\frac{\partial}{\partial y} + i\eta_1\ \lambda^2\right) V\ , \qquad h\,\frac{\partial V}{\partial x} = -\left(k\,\frac{\partial}{\partial y} + \frac{i}{\eta_2}\ \lambda^2\right) U \tag{4}$$

$$h\,\frac{\partial U}{\partial y} = -\left(k\,\frac{\partial}{\partial x} + i\eta_3\ \lambda^2\right) V\ , \qquad h\,\frac{\partial V}{\partial y} = \left(k\,\frac{\partial}{\partial x} + \frac{i}{\eta_4}\ \lambda^2\right) U \tag{5}$$

on y = 0 and x = 0 respectively, and serve to couple U and V unless h = 0 (incidence in a plane perpendicular to the edge), $\eta_2 = \eta_4 = 0$ or $\eta_1 = \eta_3 = \infty$.

To specify the incident field we choose

$$U^i = A\, e^{-i\lambda(xc + ys)} \quad , \qquad V^i = B\, e^{-i\lambda(xc + ys)} \tag{6}$$

where a factor $e^{-i(hz + \omega t)}$ has been suppressed. A and B are arbitrary constants and $c = \cos\alpha$, $s = \sin\alpha$. Since a plane wave cannot propagate parallel to the surfaces, it is necessary to assume that $c, s > 0$ implying $0 < \alpha < \pi/2$, and $\lambda \neq 0$. The task is to find the total (incident plus scattered) fields in accordance with (6) and the boundary conditions (4) and (5).

PLANE WAVE SOLUTION

We first seek a plane wave solution in the form

$$U = A\, e^{-i\lambda(xc + ys)} + A_1\, e^{-i\lambda(xc - ys)} + A_2\, e^{i\lambda(xc - ys)} + A_3\, e^{i\lambda(xc + ys)} \tag{7}$$

$$V = B\, e^{-i\lambda(xc + ys)} + B_1\, e^{-i\lambda(xc - ys)} + B_2\, e^{i\lambda(xc - ys)} + B_3\, e^{i\lambda(xc + ys)} \tag{8}$$

as suggested by the image method. The last three terms in each of (7) and (8) constitute the scattered field, and the unknown amplitudes A_i and B_i, $i = 1, 2, 3$, must be chosen to satisfy the boundary conditions (4) and (5). From the conditions on $y = 0$

$$hc\,(A + A_1) = (ks - \lambda\eta_1)\, B - (ks + \lambda\eta_1)\, B_1 \tag{9}$$

$$hc\,(A_2 + A_3) = -\,(ks - \lambda\eta_1)\, B_2 + (ks + \lambda\eta_1)\, B_3 \tag{10}$$

$$hc\,(B + B_1) = -\,(ks - \lambda/\eta_2)\, A + (ks + \lambda/\eta_2)\, A_1 \tag{11}$$

$$hc\,(B_2 + B_3) = (ks - \lambda/\eta_2)\, A_2 - (ks + \lambda/\eta_2)\, A_3\ . \tag{12}$$

Similarly, from the conditions on $x = 0$

$$hs\ (A + A_2) = -\ (kc - \lambda n_3)\ B + (kc + \lambda n_3)\ B_2 \tag{13}$$

$$hs\ (A_1 + A_3) = (kc - \lambda n_3)\ B_1 - (kc + \lambda n_3)\ B_3 \tag{14}$$

$$hs\ (B + B_2) = (kc - \lambda/n_4)\ A - (kc + \lambda/n_4)\ A_2 \tag{15}$$

$$hs\ (B_1 + B_3) = -\ (kc - \lambda/n_4)\ A_1 + (kc + \lambda/n_4)\ A_3 \tag{16}$$

yielding a total of 8 equations for the 6 unknowns A_i and B_i. From (9) and (11) A_1 and B_1 can be expressed in terms of A and B:

$$\Gamma A_1 = -\ \{\Gamma - 2ks\ (ks + \lambda n_1)\}\ A + 2\ hksc\ B \tag{17}$$

$$\Gamma B_1 = -\ 2\ hksc\ A - \{\Gamma - 2ks\ (ks + \lambda/n_2)\}\ B \tag{18}$$

where

$$\Gamma = h^2\ c^2 + (ks + \lambda n_1)(ks + \lambda/n_2)\ . \tag{19}$$

In addition, from (13) and (15)

$$\Gamma'\ A_2 = -\ \{\Gamma' - 2kc\ (kc + \lambda n_3)\}\ A - 2\ hksc\ B \tag{20}$$

$$\Gamma'\ B_2 = 2\ hksc\ A - \{\Gamma' - 2kc\ (kc + \lambda/n_4)\}\ B \tag{21}$$

where

$$\Gamma' = h^2\ s^2 + (kc + \lambda n_3)(kc + \lambda/n_4) \tag{22}$$

completing the specification of all the amplitudes save A_3 and B_3.

From (10) and (12), however, A_3 and B_3 can be expressed in terms of A_2 and B_2, and using then (20) and (21), we have

$$\begin{aligned}\Gamma\Gamma'\ A_3 = &\Big[\{\Gamma - 2ks\ (ks + \lambda n_1)\}\{\Gamma' - 2kc\ (kc + \lambda n_3)\} - 4\ h^2k^2s^2c^2\Big]A \\ &+ 2\ hksc\Big[\Gamma - 2ks\ (ks + \lambda n_1) + \Gamma' - 2kc\ (kc + \lambda/n_4)\Big]B\end{aligned} \tag{23}$$

$$\Gamma\Gamma' B_3 = -2\,hksc\Big[\Gamma - 2ks\,(ks + \lambda/\eta_2) + \Gamma' - 2kc\,(kc + \lambda\eta_3)\Big]A$$

$$+\Big[\{\Gamma - 2ks\,(ks + \lambda/\eta_2)\}\{\Gamma' - 2kc\,(kc + \lambda/\eta_4)\} - 4\,h^2k^2s^2c^2\Big]B. \tag{24}$$

Also, from (14) and (16) with (17) and (18)

$$\Gamma\Gamma' A_3 = \Big[\{\Gamma - 2ks\,(ks + \lambda\eta_1)\}\{\Gamma' - 2kc\,(kc + \lambda\eta_3)\} - 4\,h^2k^2s^2c^2\Big]A$$

$$-2\,hksc\Big[\Gamma - 2ks\,(ks + \lambda/\eta_2) + \Gamma' - 2kc\,(kc + \lambda\eta_3)\Big]B \tag{25}$$

$$\Gamma\Gamma' B_3 = 2\,hksc\Big[\Gamma - 2ks\,(ks + \lambda\eta_1) + \Gamma' - 2kc\,(kc + \lambda/\eta_4)\Big]A$$

$$+\Big[\{\Gamma - 2ks\,(ks + \lambda/\eta_2)\}\{\Gamma' - 2kc\,(kc + \lambda/\eta_4)\} - 4\,h^2k^2s^2c^2\Big]B \tag{26}$$

and the results are not necessarily equivalent. Taking first the expressions (23) and (25) for $\Gamma\Gamma' A_3$, the coefficients of A are identical but those of B can be written as

$$-2\,hksc\Big[k\lambda\,\{s\,(\eta_1 - 1/\eta_2) - c\,(\eta_3 - 1/\eta_4)\} \mp \lambda^2\Big(\frac{\eta_1}{\eta_2} + \frac{\eta_3}{\eta_4} - 1\Big)\Big] \tag{27}$$

with the upper and lower signs for (23) and (25) respectively. Since λ, s, c $\neq 0$, the coefficients are the same if and only if h = 0 corresponding to incidence in a plane perpendicular to the edge, or

$$\frac{\eta_1}{\eta_2} + \frac{\eta_3}{\eta_4} - 1 = 0\ . \tag{28}$$

Similarly, in the expressions (24) and (26) for $\Gamma\Gamma' B_3$, the coefficients of B are identical but those of A are now given by (27), leading again to the requirement that (28) be satisfied to make them the same. Equation (28) is the impedance compatibility relation obtained by Dybdal et al (1971) as the condition for the existence of discrete modes in a rectangular waveguide whose opposite walls have the same anisotropic impedances. It is equivalent to

$$\underline{E} \cdot \underline{H} = 0$$

at the edge, and if it is satisfied

$$\Gamma\Gamma' A_3 = \left[\{\Gamma - 2ks\,(ks + \lambda\eta_1)\}\{\Gamma' - 2kc\,(kc + \lambda\eta_3)\} - 4\,h^2k^2s^2c^2\right]A$$
$$- 2hk^2\,\lambda\,sc\,\{s\,(\eta_1 - 1/\eta_2) - c\,(\eta_3 - 1/\eta_4)\}\,B \qquad (29)$$

$$\Gamma\Gamma' B_3 = - 2hk^2\,\lambda\,sc\,\{s\,(\eta_1 - 1/\eta_2) - c\,(\eta_3 - 1/\eta_4)\}\,A$$
$$+ \left[\{\Gamma - 2ks\,(ks + \lambda/\eta_2)\}\{\Gamma' - 2kc\,(kc + \lambda/\eta_4)\} - 4\,h^2k^2s^2c^2\right]B. \qquad (30)$$

The expressions (7) and (8) for U and V then constitute a valid solution with the amplitudes A_i and B_i, i = 1, 2, 3, defined by (17), (18), (20), (21), (29) and (30). Since U and V and, hence, E_z and H_z are non-zero at the edge, the components of the electric and magnetic currents perpendicular to the edge are continuous from one face to the other, but as a result of the boundary conditions the component of the electric current parallel to the edge is discontinuous unless $\eta_2 = \eta_4$ and the same component of the magnetic current is discontinuous unless $\eta_1 = \eta_3$.

Our final remarks are for the case when the impedances are small. If

$$\eta_i = \varepsilon_i \eta \qquad i = 1, 2, 3, 4$$

where the ε_i are of order unity and $\eta \ll 1$, the discrepancies between (23) and (25) and (24) and (26) are $O(\eta^2)$. It follows that to $O(\eta)$ our plane wave solution is valid regardless of the compatibility relation (28), and it is now of interest to consider the form taken by the next higher order term for small but arbitrary impedances.

SMALL IMPEDANCE SOLUTION

To obtain a solution which holds regardless of the relation (28), we assume the following series expansions for U and V:

$$U(x, y) = \sum_{m=0}^{\infty} \eta^m U_m(x, y), \qquad V(x, y) = \sum_{m=0}^{\infty} \eta^m V_m(x, y) \tag{31}$$

On substituting into the boundary conditions (4) and (5) and equating coefficients of like powers of η, we find

$$h \frac{\partial U_m}{\partial x} - k \frac{\partial V_m}{\partial y} = i \varepsilon_1 \lambda^2 V_{m-1}, \quad i \lambda^2 U_m = - \varepsilon_2 \left(k \frac{\partial V_{m-1}}{\partial y} + h \frac{\partial V_{m-1}}{\partial x} \right) \quad (y = 0)$$

$$h \frac{\partial U_m}{\partial y} + k \frac{\partial V_m}{\partial x} = - i \varepsilon_3 \lambda^2 V_{m-1}, \quad i \lambda^2 U_m = - \varepsilon_4 \left(k \frac{\partial U_{m-1}}{\partial x} - h \frac{\partial V_{m-1}}{\partial y} \right) \quad (x = 0)$$

for $m \geq 0$ with $U_{-1}, V_{-1} \equiv 0$, and these can be expressed as

$$U_m(x, y) = \begin{cases} f_m^{(1)}(x) & y = 0, x \geq 0 \\ f_m^{(2)}(y) & x = 0, y \geq 0 \end{cases} \tag{32}$$

$$\frac{\partial}{\partial n} V_m (x, y) = \begin{cases} g_m^{(1)}(x) & y = 0,\ x \geq 0 \\ g_m^{(2)}(y) & x = 0,\ y \geq 0 \end{cases} \tag{33}$$

where n is the outward normal and

$$f_m^{(1)}(x) = \frac{i\varepsilon_2}{\lambda^2} \left(k \frac{U_{m-1}}{\partial y} + h \frac{V_{m-1}}{\partial x} \right) \Bigg|_{y = 0} \tag{34}$$

$$f_m^{(2)}(y) = \frac{i\varepsilon_4}{\lambda^2} \left(k \frac{\partial U_{m-1}}{\partial x} - h \frac{\partial V_{m-1}}{\partial y} \right) \Bigg|_{x = 0} \tag{35}$$

$$g_m^{(1)}(x) = - \frac{i}{k\lambda^2} \left(\varepsilon_1 \lambda^4 V_{m-1} - \varepsilon_2 h^2 \frac{\partial^2 V_{m-1}}{\partial x^2} - \varepsilon_2 kh \frac{\partial^2 U_{m-1}}{\partial x \partial y} \right) \Bigg|_{y = 0} \tag{36}$$

$$g_m^{(2)}(y) = - \frac{i}{k\lambda^2} \left(\varepsilon_3 \lambda^4 V_{m-1} - \varepsilon_4 h^2 \frac{\partial^2 V_{m-1}}{\partial x^2} - \varepsilon_4 kh \frac{\partial^2 U_{m-1}}{\partial x \partial y} \right) \Bigg|_{x = 0} . \tag{37}$$

The task has now been reduced to the solution of separate boundary value problems for U_m and V_m.

Given a scalar field $\psi(x, y)$ satisfying the wave equation (3), the field at any point inside and on a closed contour L can be written as

$$\psi(x, y) = \int_L \left(\psi \frac{\partial G}{\partial n'} - G \frac{\partial \psi}{\partial n'} \right) ds' \tag{38}$$

where G is a two-dimensional Green's function. In the case $\psi = U_m$ it is convenient to choose $G = G_s$:

$$G_s = \frac{i}{4} \left\{ H_o^{(1)}(\lambda R_1) - H_o^{(1)}(\lambda R_2) - H_o^{(1)}(\lambda R_3) + H_o^{(1)}(\lambda R_4) \right\}$$

with

$$R_{1,\,2} = \left\{(x - x')^2 + (y \mp y')^2\right\}^{1/2}$$

$$R_{3,\,4} = \left\{(x + x')^2 + (y \mp y')^2\right\}^{1/2}$$

where $H_o^{(1)}$ is the Hankel function of the first kind of order zero, so that G_s vanishes on the surfaces $x = 0$ and $y = 0$. On applying (38) to the region bounded by the lines $y = 0$ ($x \geq 0$), $x = 0$ ($y \geq 0$) and a quarter circle at infinity, and imposing the boundary conditions (34) and (35), we have

$$U_m\,(x,\,y) = \int_o^\infty f_m^{(1)}(x')\,\frac{\partial G_s}{\partial y'}\bigg|_{y'=0} dx' + \int_o^\infty f_m^{(2)}(y')\,\frac{\partial G_s}{\partial x'}\bigg|_{x'=0} dy'$$

since the quarter circle yields no contribution. Hence

$$U_m\,(x,\,y) =$$

$$- \frac{i}{2}\int_o^\infty f_m^{(1)}(x')\left\{\frac{\partial}{\partial y} H_o^{(1)}\left[\lambda\sqrt{(x - x')^2 + y^2}\right] - \frac{\partial}{\partial y} H_o^{(1)}\left[\lambda\,\sqrt{(x + x')^2 + y^2}\right]\right\}dx'$$

$$- \frac{i}{2}\int_o^\infty f_m^{(2)}(y')\left\{\frac{\partial}{\partial x} H_o^{(1)}\left[\lambda\sqrt{(y - y')^2 + x^2}\right] - \frac{\partial}{\partial x} H_o^{(1)}\left[\lambda\,\sqrt{(y + y')^2 + x^2}\right]\right\}dy' \tag{39}$$

from which U_m can be computed knowing $f_m^{(1)}(x)$ and $f_m^{(2)}(y)$.

For the partial field V_m the appropriate choice of G is $G = G_h$ where

$$G_h = \frac{i}{4}\left\{H_o^{(1)}(\lambda R_1) + H_o^{(1)}(\lambda R_2) + H_o^{(1)}(\lambda R_3) + H_o^{(1)}(\lambda R_4)\right\},$$

and using this and proceeding in an analogous manner to the above, we find

$V_m(x, y) =$

$$- \frac{i}{2} \int_0^\infty g_m^{(1)}(x') \left\{ H_o^{(1)} \left[\lambda\sqrt{(x - x')^2 + y^2} \right] + H_o^{(1)} \left[\lambda\sqrt{(x + x')^2 + y^2} \right] \right\} dx'$$
$$- \frac{i}{2} \int_0^\infty g_m^{(2)}(y') \left\{ H_o^{(1)} \left[\lambda\sqrt{(y - y')^2 + x^2} \right] + H_o^{(1)} \left[\lambda\sqrt{(y + y')^2 + x^2} \right] \right\} dy' \tag{40}$$

which specifies V_m in terms of $g_m^{(1)}(x)$ and $g_m^{(2)}(y)$.

The incident plane wave is given by (6), and since the boundary conditions on the zero order fields are

$$U_o \, , \frac{\partial V_o}{\partial y} = 0 \qquad \text{on } y = 0, \ x \geq 0$$

$$U_o \, , \frac{\partial V_o}{\partial x} = 0 \qquad \text{on } x = 0, \ y \geq 0$$

it follows immediately that

$$U_o(x, y) = A \left\{ e^{-i\lambda(xc + ys)} - e^{-i\lambda(xc - ys)} - e^{i\lambda(xc - ys)} + e^{i\lambda(xc + ys)} \right\} \tag{41}$$

$$V_o(x, y) = B \left\{ e^{-i\lambda(xc + ys)} + e^{-i\lambda(xc - ys)} + e^{i\lambda(xc - ys)} + e^{i\lambda(xc + ys)} \right\} . \tag{42}$$

We can now determine the boundary values for the first order fields. From (34) and (35) with m = 1,

$$f_1^{(1)}(x) = - \frac{4i\varepsilon_2}{\lambda} (ks\ A + hc\ B) \sin \lambda xc \ ,$$

$$f_1^{(2)}(y) = - \frac{4i\varepsilon_4}{\lambda} (kc\ A - hs\ B) \sin \lambda ys \ ,$$

and when substituted into (39), we have (see A.3)

$$U_1\ (x, y) = -\frac{2\varepsilon_2}{\lambda}(ks\ A + hc\ B)\left\{e^{i\lambda(xc + ys)} - e^{-i\lambda(xc - ys)}\right\}$$

$$-\frac{2\varepsilon_4}{\lambda}(kc\ A - hs\ B)\left\{e^{i\lambda(xc + ys)} - e^{\ i\lambda(xc - ys)}\right\}. \qquad (43)$$

Similarly, for V_1 ,

$$g_1^{(1)}(x) = -\frac{4i}{k}\left\{\varepsilon_2\ khsc\ A + (\varepsilon_1\ \lambda^2 + \varepsilon_2\ h^2c^2)\ B\right\}\cos\lambda xc\ ,$$

$$g_1^{(2)}(y) = \frac{4i}{k}\left\{\varepsilon_4\ khsc\ A - (\varepsilon_3\ \lambda^2 + \varepsilon_4\ h^2s^2)\ B\right\}\cos\lambda ys\ ,$$

and when substituted into (40), the result is (see A.5)

$$V_1\ (x, y) = -\frac{2}{k\lambda s}\left\{\varepsilon_2\ khsc\ A + (\varepsilon_1\ \lambda^2 + \varepsilon_2\ h^2c^2)\ B\right\}\left\{e^{i\lambda(xc + ys)} + e^{-i\lambda(xc - ys)}\right\}$$

$$+\frac{2}{k\lambda c}\left\{\varepsilon_4\ khsc\ A - (\varepsilon_3\ \lambda^2 + \varepsilon_4\ h^2s^2)\ B\right\}\left\{e^{i\lambda(xc + ys)} + e^{i\lambda(xc - ys)}\right\}. \qquad (44)$$

These serve to specify the boundary values for the second order fields. From (34) and (35) with m = 2,

$$f_2^{(1)}(x) = \frac{4i\varepsilon_2}{k\lambda^2 s}\left\{k\varepsilon_2 s\ (k^2 - \lambda^2c^2)\ A + hc\ [\varepsilon_1\ \lambda^2 + \varepsilon_2\ (k^2 - \lambda^2c^2)]\ B\right\}\sin\lambda xc$$

$$+\frac{4\varepsilon_2}{k}\left\{\varepsilon_4\ ksc\ A + h\ (\varepsilon_3 - \varepsilon_4 s^2)\ B\right\}e^{i\lambda xc}\ ,$$

$$f_2^{(2)}(y) = \frac{4i\varepsilon_2}{k\lambda^2 c}\left\{k\varepsilon_4 c\ (k^2 - \lambda^2s^2)\ A - hs\ [\varepsilon_3\ \lambda^2 + \varepsilon_4(k^2 - \lambda^2s^2)]\ B\right\}\sin\lambda ys$$

$$+\frac{4\varepsilon_4}{k}\left\{\varepsilon_2\ ksc\ A - h\ (\varepsilon_1 - \varepsilon_2 c^2)\ B\right\}e^{i\lambda ys}\ ,$$

giving

$$U_2 (x, y) =$$

$$= \frac{2\varepsilon_2}{k\lambda^2 s} \left\{ k\varepsilon_2 s (k^2 - \lambda^2 c^2) A + hc [\varepsilon_1 \lambda^2 + \varepsilon_2 (k^2 - \lambda^2 c^2)] B \right\} \left\{ e^{i\lambda(xc + ys)} - e^{-i\lambda(xc - ys)} \right\} + \frac{2\varepsilon_4}{k\lambda^2 c} \left\{ k\varepsilon_4 c (k^2 - \lambda^2 s^2) A - hs [\varepsilon_3 \lambda^2 + \varepsilon_4 (k^2 - \lambda^2 s^2)] B \right\}$$
$$\left\{ e^{i\lambda(xc + ys)} - e^{i\lambda(xc - ys)} \right\} + \frac{4\varepsilon_2}{k} \left\{ \varepsilon_4 ksc A + h (\varepsilon_3 - \varepsilon_4 s^2) B \right\} J_1 (x, y; \alpha)$$
$$+ \frac{4\varepsilon_4}{k} \left\{ \varepsilon_2 ksc A - h (\varepsilon_1 - \varepsilon_2 c^2) B \right\} J_1 (y, x; \tfrac{\pi}{2} - \alpha)$$

where the function J_1 is defined in the Appendix. As shown there, however,

$$J_1 (y, x; \tfrac{\pi}{2} - \alpha) = e^{i\lambda(xc + ys)} - J_1 (x, y; \alpha) ,$$

and U_2 can therefore be written as

$$U_2 (x, y) =$$

$$= \frac{2\varepsilon_2}{k\lambda^2 s} \left\{ k\varepsilon_2 s (k^2 - \lambda^2 c^2) A + hc [\varepsilon_1 \lambda^2 + \varepsilon_2 (k^2 - \lambda^2 c^2)] B \right\} \left\{ e^{i\lambda(xc + ys)} - e^{-i\lambda(xc - ys)} \right\} + \frac{2\varepsilon_4}{k\lambda^2 c} \left\{ k\varepsilon_4 c (k^2 - \lambda^2 s^2) A - hs [\varepsilon_3 \lambda^2 + \varepsilon_4 (k^2 - \lambda^2 s^2)] B \right\}$$
$$\left\{ e^{i\lambda(xc + ys)} - e^{i\lambda(xc - ys)} \right\} + \frac{4\varepsilon_4}{k} \left\{ \varepsilon_2 ksc A - h (\varepsilon_1 - \varepsilon_2 c^2) B \right\} e^{i\lambda(xc + ys)}$$
$$+ \frac{4h}{k} (\varepsilon_1 \varepsilon_4 + \varepsilon_2 \varepsilon_3 - \varepsilon_2 \varepsilon_4) B J_1 (x, y; \alpha) . \qquad (45)$$

Similarly

$$g_2^{(1)}(x) =$$

$$\frac{4i}{k^2\lambda s}\left\{\varepsilon_2\ khsc\ [\varepsilon_1\ \lambda^2 + \varepsilon_2\ (k^2 - \lambda^2c^2)]\ A + [(\varepsilon_1\ \lambda^2 + \varepsilon_2\ h^2c^2)^2 + (\varepsilon_2\ khsc)^2]\ B\right\}$$

$$\cdot \cos \lambda\ xc$$

$$- \frac{4i\lambda}{k^2c}\left\{\varepsilon_4\ khsc\ (\varepsilon_1 - \varepsilon_2c^2)\ A - [(\varepsilon_3 - \varepsilon_4s^2)(\varepsilon_1\ \lambda^2 + \varepsilon_2\ h^2c^2) + \varepsilon_1\varepsilon_4\ k^2s^2]\ B\right\}$$

$$\cdot\ e^{i\lambda\ xc}\ ,$$

$$g_2^{(2)}(y) =$$

$$- \frac{4i}{k^2\lambda c}\left\{\varepsilon_4\ khsc\ [\varepsilon_3\ \lambda^2 + \varepsilon_4\ (k^2 - \lambda^2s^2)]\ A - [(\varepsilon_3\ \lambda^2 + \varepsilon_4\ h^2s^2)^2 + (\varepsilon_4\ khsc)^2]\ B\right\}$$

$$\cdot \cos \lambda\ ys$$

$$+ \frac{4i\lambda}{k^2s}\left\{\varepsilon_2\ khsc\ (\varepsilon_3 - \varepsilon_4s^2)A + [(\varepsilon_3 - \varepsilon_4s^2)(\varepsilon_1\ \lambda^2 + \varepsilon_2\ h^2c^2) + \varepsilon_1\varepsilon_4\ k^2s^2]\ B\right\}$$

$$\cdot\ e^{i\lambda\ ys}\ .$$

On substituting these into (40) and using (A.4), (A.12), (A.14) and (A.15), we find

$$V_2\ (x,\ y) =$$

$$\frac{2}{k^2\lambda^2s^2}\left\{\varepsilon_2\ khsc\ [\varepsilon_1\ \lambda^2 + \varepsilon_2\ (k^2 - \lambda^2c^2)]\ A + [(\varepsilon_1\ \lambda^2 + \varepsilon_2\ h^2c^2)^2 + (\varepsilon_2\ khsc)^2]\ B\right\}$$

$$\cdot \left\{e^{i\lambda(xc\ +\ ys)} + e^{-i\lambda(xc\ -\ ys)}\right\}$$

waveguide having two adjacent walls imperfectly conducting.

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$$- \frac{2}{k^2\lambda^2c^2}\left\{\varepsilon_4 \, khsc \, [\varepsilon_3 \lambda^2 + \varepsilon_4 (k^2 - \lambda^2 s^2)] A - [(\varepsilon_3 \lambda^2 + \varepsilon_4 h^2 s^2)^2 + (\varepsilon_4 \, khsc)^2] B\right\}$$

$$\cdot \left\{e^{i\lambda(xc + ys)} + e^{i\lambda(xc - ys)}\right\}$$

$$+ \frac{4}{k^2 sc}\left\{\varepsilon_2 \, khsc \, (\varepsilon_3 - \varepsilon_4 s^2) A + [(\varepsilon_3 - \varepsilon_4 s^2)(\varepsilon_1 \lambda^2 + \varepsilon_2 h^2 c^2) + \varepsilon_1 \varepsilon_4 k^2 s^2] B\right\}$$

$$\cdot \, e^{i\lambda(xc + ys)}$$

$$- \frac{4h}{k} (\varepsilon_1\varepsilon_4 + \varepsilon_2\varepsilon_3 - \varepsilon_2\varepsilon_4) A J_2 (x, y; \alpha) \qquad (46)$$

with J_2 defined in the Appendix.

Since A = B = 0 implies a null solution, the last term in each of (45) and (46) vanishes only if the impedance compatibility relation (28) is satisfied. U_2 and V_2 then consist of three plane wave contributions, and the resulting expressions for

$$U = U_o + \eta U_1 + \eta^2 U_2 , \qquad V = V_o + \eta V_1 + \eta^2 V_2$$

are identical to the expansions through terms $O(\eta^2)$ of the solution derived in the previous section. If desired, higher order terms can also be obtained, and involve only the plane waves already present in U_2 and V_2. If (28) is not satisfied, however, the character of the solution changes considerably. U_2 and V_2 now contain terms proportional to J_1 and J_2 respectively, and it is of interest to examine the solution in this more general case.

DISCUSSION

The functions $J_1 (x, y; \alpha)$ and $J_2 (x, y; \alpha)$ satisfy respectively the integral equations

$$f\ (x,\ y) = \int_{\ell} f\ (x',\ y')\ \frac{\partial G_s}{\partial n'}\ ds'$$

and

$$g\ (x,\ y) = - \int_{\ell} G_h\ \frac{\partial}{\partial n'}\ g\ (x',\ y')\ ds'$$

where ℓ is the contour consisting of the straight line segments $y = 0,\ x \geq 0$ and $x = 0,\ y \geq 0$. Each can be expressed (see Appendix) in terms of the function

$$F\ (\tau\ ,\ \psi) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\sin\psi}{\cosh t + \cos\psi}\ e^{i\tau\cosh t}\ dt \tag{47}$$

which occurs in Kottler's (1923) solution for the diffraction of a plane wave by a perfectly 'black' half plane. The integral is absolutely convergent if Im. $\tau > 0$ and $-\pi < \psi < \pi$, and has been discussed at length by Watson (1938), Copson and Ferrar (1938) and Erdélyi (1938a, b). It is similar to the Fresnel integral and can be expanded in a series of 'cut' Hankel functions (Copson and Ferrar, 1938). In addition,

$$F\ (\tau\ ,\ \psi) = e^{-i\tau\cos\psi}\left\{\frac{\psi}{2\pi} - \frac{1}{4}\sin\psi \int_0^{\tau} H_0^{(1)}(t)\ e^{it\cos\psi}\ dt\right\} \tag{48}$$

(Erdélyi, 1938a), constituting a finite integral representation, but no analytic evaluation of the integral has yet been achieved. For $\tau >> 1$ the stationary phase approximation to (47) is,

$$F\ (\tau\ ,\ \psi) \sim \frac{1}{4}\sqrt{\frac{2}{\pi\tau}}\ e^{i(\tau + \pi/4)}\ \tan\frac{\psi}{2}$$

and for $\tau << 1$ (48) shows

$$F\ (\tau\ ,\ \psi) = \frac{\psi}{2\pi} + 0\ (\tau \log \tau).$$

From the expressions (A.9) and (A.13) for J_1 and J_2 respectively in terms of $F(\tau, \psi)$, it follows immediately that on the line segment $y = 0$, $x \geq 0$

$$J_1 = e^{i\lambda x \cos \alpha} , \qquad \frac{\partial J_2}{\partial y} = i\lambda \sin \alpha \, e^{i\lambda x \cos}$$

whereas on the segment $x = 0$, $y \geq 0$

$$J_1 = 0 \quad , \qquad \frac{\partial J_2}{\partial x} = 0.$$

J_1 and J_2 are also closely related and if $J_1 (x, y; \alpha) = K (\theta , \alpha)$ where θ is the cylindrical polar coordinate, $J_2 (x, y; \alpha) = K (\pi/2 - \alpha, \pi/2 - \theta)$. For $\lambda\rho >> 1$ and θ bounded away from α

$$J_1 (x, y; \alpha) \sim e^{i\lambda\rho \cos (\alpha - \theta)} H (\alpha - \theta) - \sqrt{\frac{2}{\pi\lambda\rho}} e^{i (\lambda\rho + \pi/4)} \frac{\sin 2\theta}{\cos 2\theta - \cos 2\alpha} ,$$

$$J_2 (x, y; \alpha) \sim e^{i\lambda\rho \cos (\alpha - \theta)} H (\alpha - \theta) - \sqrt{\frac{2}{\pi\lambda\rho}} e^{i (\lambda\rho + \pi/4)} \frac{\sin 2\theta}{\cos 2\theta - \cos 2\alpha} ,$$

where H is the unit step function, showing that at large distances from the edge, both functions have the appearance of an edge wave. For $\lambda\rho << 1$, however,

$$J_1 (x, y; \alpha) \sim 1 - 2\theta/\pi , \qquad J_2 (x, y; \alpha) \sim 2\alpha/\pi .$$

The partial field V_2 is therefore continuous from one surface of the wedge to the other, but U_2 has a finite jump discontinuity.

Partial fields U_m and V_m of higher order than the second can be obtained by the same method used to determine U_2 and V_2, and these turn out to be singular at the edge, with the order of the singularity increasing with m. They also involve functions which, though similar to J_1 and J_2, become more complicated as m increases, and for this reason it does not seem likely that the field expansions can be summed to arrive at an exact closed form solution of the wedge problem. Nevertheless, it is evident that like the partial fields U_m and V_m for $m \geq 2$, the exact solution must have a continuous angular spectrum.

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APPENDIX: SOME INTEGRAL EVALUATIONS

There are two types of integral that arise in connection with the partial fields U_1 and V_1. The first is

$$I_1 = \int_0^\infty \sin(\lambda\, x'\, c) \left\{ \frac{\partial}{\partial y} H_o^{(1)}(\lambda\sqrt{(x - x')^2 + y^2}) - \frac{\partial}{\partial x} H_o^{(1)}(\lambda\sqrt{(x + x')^2 + y^2}) \right\} dx' \tag{A.1}$$

which can be written as

$$I_1 = \int_{-\infty}^\infty \sin(\lambda\, x'\, c) \frac{\partial}{\partial y} H_o^{(1)}(\lambda\sqrt{(x - x')^2 + y^2}) \; dx'.$$

We now insert the Fourier integral representation

$$H_o^{(1)}(\lambda\sqrt{(x - x')^2 + y^2}) = \frac{1}{\pi}\int_{-\infty}^\infty e^{i\omega\,(x - x') + iy\sqrt{\lambda^2 - \omega^2}} \frac{d\omega}{\sqrt{\lambda^2 - \omega^2}} \tag{A.2}$$

valid for $y \geq 0$, where λ is assumed to have a small positive imaginary part. On reversing the order of integration and using the fact that

$$\int_{-\infty}^\infty e^{-i\,(\omega \mp \lambda c)\,x'} dx' = 2\pi\,\delta\,(\omega \mp \lambda c)$$

where δ is the Dirac delta function, we have

$$I_1 = e^{i\lambda\, ys}\,(e^{i\lambda\, xc} - e^{-i\lambda\, xc})\ . \tag{A.3}$$

The second integral is

$$I_2 = \lambda s\int_0^\infty \cos(\lambda\, x'\, c)\left\{ H_o^{(1)}(\lambda\sqrt{(x - x')^2 + y^2}) + H_o^{(1)}(\lambda\sqrt{(x + x')^2 + y^2}\right\} dx' \tag{A.4}$$

and in a similar manner it can be shown that

$$I_2 = e^{i\lambda\, ys} \left(e^{i\lambda\, xc} + e^{-i\lambda\, xc}\right). \tag{A.5}$$

For integrals that differ from the above in having only single exponentials in the integrands the evaluation is more difficult. Consider

$$J_1 (x, y; \alpha) =$$

$$-\frac{i}{2}\int_0^\infty e^{i\lambda\, x' \cos\alpha} \left\{ \frac{\partial}{\partial y} H_0^{(1)}\left(\lambda\sqrt{(x - x')^2 + y^2}\right) - \frac{\partial}{\partial y} H_0^{(1)}\left(\lambda\sqrt{(x + x')^2 + y^2}\right) \right\} dx' \tag{A.6}$$

where $0 < \alpha < \pi$. If we again insert the Fourier integral representation (A.2), the x' integration can be carried out immediately to give

$$\int_0^\infty e^{i\,(\lambda \cos\alpha \mp \omega)\, x'}\, dx' = \frac{i}{\lambda \cos\alpha \mp \omega}$$

and hence

$$J_1 (x, y; \alpha) = -\frac{i}{\pi}\int_{-\infty}^{\infty} \frac{\omega}{\omega^2 - \lambda^2 \cos^2\alpha}\, e^{i\omega x + iy\sqrt{\lambda^2 - \omega^2}}\, d\omega.$$

We now make the substitution $\omega = \lambda \cos \gamma$ and write

$$x = \rho \cos\theta\,, \qquad y = \rho \sin\theta$$

with $0 \leq \theta \leq \pi/2$, so that

$$J_1 (x, y; \alpha) = -\frac{i}{\pi}\int_C \frac{\sin\gamma \cos\gamma}{\cos^2\gamma - \cos^2\alpha}\, e^{i\lambda\rho \cos(\gamma - \theta)}\, d\gamma$$

where C is the path shown in Figure A-1. The stationary phase point is $\gamma = \theta$, and in a deformation of the path into the stationary phase contour $S(\theta)$ the pole at $\gamma = \alpha$ is captured if $\theta < \alpha$. Hence

$$J_1 (x, y; \alpha) = e^{i\lambda\rho \cos(\alpha - \theta)} H(\alpha - \theta) - \frac{i}{\pi} \int_{S(0)} \frac{\sin(\gamma + \theta)\cos(\gamma + \theta)}{\cos^2(\gamma + \theta) - \cos^2\alpha}$$

$$\cdot e^{i\lambda\rho\cos\gamma} d\gamma \tag{A.7}$$

where H denotes the unit step function.

The path S(0) is symmetrical about the origin of the γ plane, and the method employed by Clemmow (1951, p. 293) can now be used to reduce the integral to a more elementary form. Replacing γ by $-\gamma$ and then adding the resulting integral to that in (A.7), we find after some manipulation

$$J_1 (x, y; \alpha) = e^{i\lambda\rho \cos(\alpha - \theta)} H(\alpha - \theta) + \frac{i}{4\pi} \int_{S(0)} \left\{ \frac{\sin(\theta + \alpha)}{\cos\gamma - \cos(\theta + \alpha)} \right.$$

$$\left. - \frac{\sin(\theta + \alpha)}{\cos\gamma + \cos(\theta + \alpha)} - \frac{\sin(\theta - \alpha)}{\cos\gamma + \cos(\theta - \alpha)} + \frac{\sin(\theta - \alpha)}{\cos\gamma - \cos(\theta - \alpha)} \right\}$$

$$\cdot e^{i\lambda\rho\cos\gamma} d\gamma\ . \tag{A.8}$$

Each of these integrals is only a special case of the Kottler function $F(\tau, \psi)$ defined in (47). A simple change in the variable of integration shows

$$F(\lambda\rho, \psi) = \frac{i}{4\pi} \int_{S(0)} \frac{\sin\psi}{\cos\gamma + \cos\psi} e^{i\lambda\rho\cos\gamma} d\gamma\ ,$$

and with the restriction $-\pi < \psi < \pi$, the first three integrals in (A.8) correspond to the identifications $\psi = \pi - \theta - \alpha$, $\theta + \alpha$ and $\theta - \alpha$ respectively. For the fourth integral $\psi = \pm\pi - \theta + \alpha$ according as $\theta \gtrless \alpha$ respectively, and since (48) implies

$$F(\lambda\rho, \pi - \theta + \alpha) - F(\lambda\rho, -\pi - \theta + \alpha) = e^{i\lambda\rho\cos(\alpha - \theta)},$$

it follows that

$$J_1\,(x,\,y;\,\alpha) =$$

$$F\,(\lambda\rho\,,\,\pi - \theta - \alpha) - F\,(\lambda\rho\,,\,\theta + \alpha) - F\,(\lambda\rho\,,\,\theta - \alpha) + F\,(\lambda\rho\,,\,\pi - \theta + \alpha). \tag{A.9}$$

On interchanging x and y in (A.6) and, at the same time, replacing α by $\pi/2 - \alpha$, we have

$$J_1\,(y,\,x;\,\pi/2 - \alpha) =$$

$$- \frac{i}{2}\int_0^{\infty} e^{i\lambda y'}\,\sin\alpha\left\{\frac{\partial}{\partial x}\,H_o^{(1)}(\lambda\sqrt{(y - y')^2 + x^2}) - \frac{\partial}{\partial x}\,H_o^{(1)}(\lambda\sqrt{(y + y')^2 + x^2})\right\} dy'$$

$$= F\,(\lambda\rho\,,\,\theta + \alpha) - F\,(\lambda\rho\,,\,\pi - \theta - \alpha) - F\,(\lambda\rho\,,\,\alpha - \theta) + F\,(\lambda\rho\,,\,\pi + \theta - \alpha) \tag{A.10}$$

and hence, from (48),

$$J_1\,(x,\,y;\,\alpha) + J_1\,(y,\,x;\,\pi/2 - \alpha) = e^{i\lambda\rho\,\cos\,(\alpha - \theta)}\;. \tag{A.11}$$

The only integral still to be evaluated is the analogue of (A.4) with the cosine factor replaced by a single exponential. If

$$J_2\,(x,\,y;\,\alpha) =$$

$$\frac{\lambda\,\sin\,\alpha}{2}\int_0^{\infty} e^{i\lambda x'}\,\cos\alpha\left\{H_o^{(1)}(\lambda\sqrt{(x - x')^2 + y^2}) + H_o^{(1)}(\lambda\sqrt{(x + x')^2 + y^2})\right\} dx', \tag{A.12}$$

insertion of the Fourier integral representation of the Hankel function gives

$J_2\ (x,\ y;\ \alpha) =$

$$- \frac{i\lambda^2}{\pi} \int_0^\infty \frac{\sin\alpha \cos\alpha}{\omega^2 - \lambda^2 \cos^2\alpha}\, e^{i\omega x + iy\sqrt{\lambda^2 - \omega^2}} \frac{d\omega}{\sqrt{\lambda^2 - \omega^2}}$$

$$= e^{i\lambda\rho \cos(\alpha - \theta)}\, H(\alpha - \theta) - \frac{i}{\pi} \int_{S(0)} \frac{\sin\alpha \cos\alpha}{\cos^2(\gamma + \theta) - \cos^2\alpha}\, e^{i\lambda\rho\cos\gamma}\, d\gamma\ ,$$

and by a process similar to that used above, we find

$J_2\ (x,\ y;\ \alpha) =$

$$e^{i\lambda\rho \cos(\alpha - \theta)}\, H(\alpha - \theta) - \frac{i}{4\pi} \int_{S(0)} \left\{ \frac{\sin(\theta + \alpha)}{\cos\gamma - \cos(\theta + \alpha)} - \frac{\sin(\theta + \alpha)}{\cos\gamma + \cos(\theta + \alpha)} \right.$$

$$\left. + \frac{\sin(\theta - \alpha)}{\cos\gamma + \cos(\theta - \alpha)} - \frac{\sin(\theta - \alpha)}{\cos\gamma - \cos(\theta - \alpha)} \right\} e^{i\lambda\rho\cos\gamma}\, d\gamma\ .$$

Hence

$J_2\ (x,\ y;\ \alpha) =$

$$- F(\lambda\rho,\ \pi - \theta - \alpha) + F(\lambda\rho,\ \theta + \alpha) - F(\lambda\rho,\ \theta - \alpha) + F(\lambda\rho,\ \pi - \theta + \alpha). \tag{A.13}$$

With θ and α replaced by $\pi/2 - \theta$ and $\pi/2 - \alpha$ respectively,

$J_2\ (y,\ x;\ \pi/2 - \alpha) =$

$$\frac{\lambda\cos\alpha}{2} \int_0^\infty e^{i\lambda y' \sin\alpha} \left\{ H_0^{(1)}\left(\lambda\sqrt{(y - y')^2 + x^2}\right) + H_0^{(1)}\left(\lambda\sqrt{(y + y')^2 + x^2}\right) \right\} dy'$$

$$= - F(\lambda\rho,\ \theta + \alpha) + F(\lambda\rho,\ \pi - \theta - \alpha) - F(\lambda\rho,\ \alpha - \theta) + F(\lambda\rho,\ \pi + \theta - \alpha)\ , \tag{A.14}$$

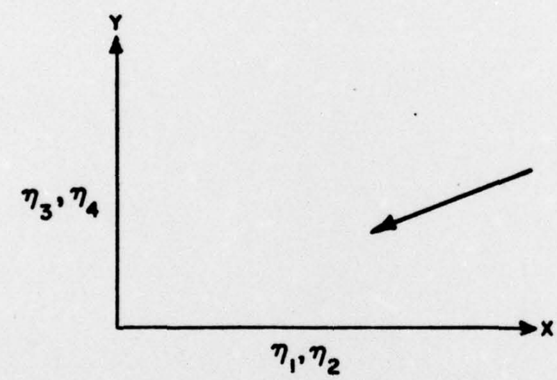
and (48) then implies

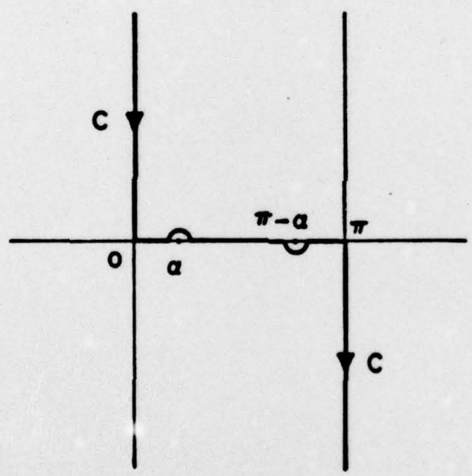
$$J_2\,(x,\, y;\, \alpha) + J_2\,(y,\, x;\, \pi/2 - \alpha) = e^{i\lambda\rho \cos(\alpha - \theta)}\,. \tag{A.15}$$

FIGURE CAPTIONS

Figure 1: The geometry.

Figure A-1: Path of integration in complex γ plane.


Y
η_3, η_4
X
η_1, η_2


C
π − a
π
0
a
C